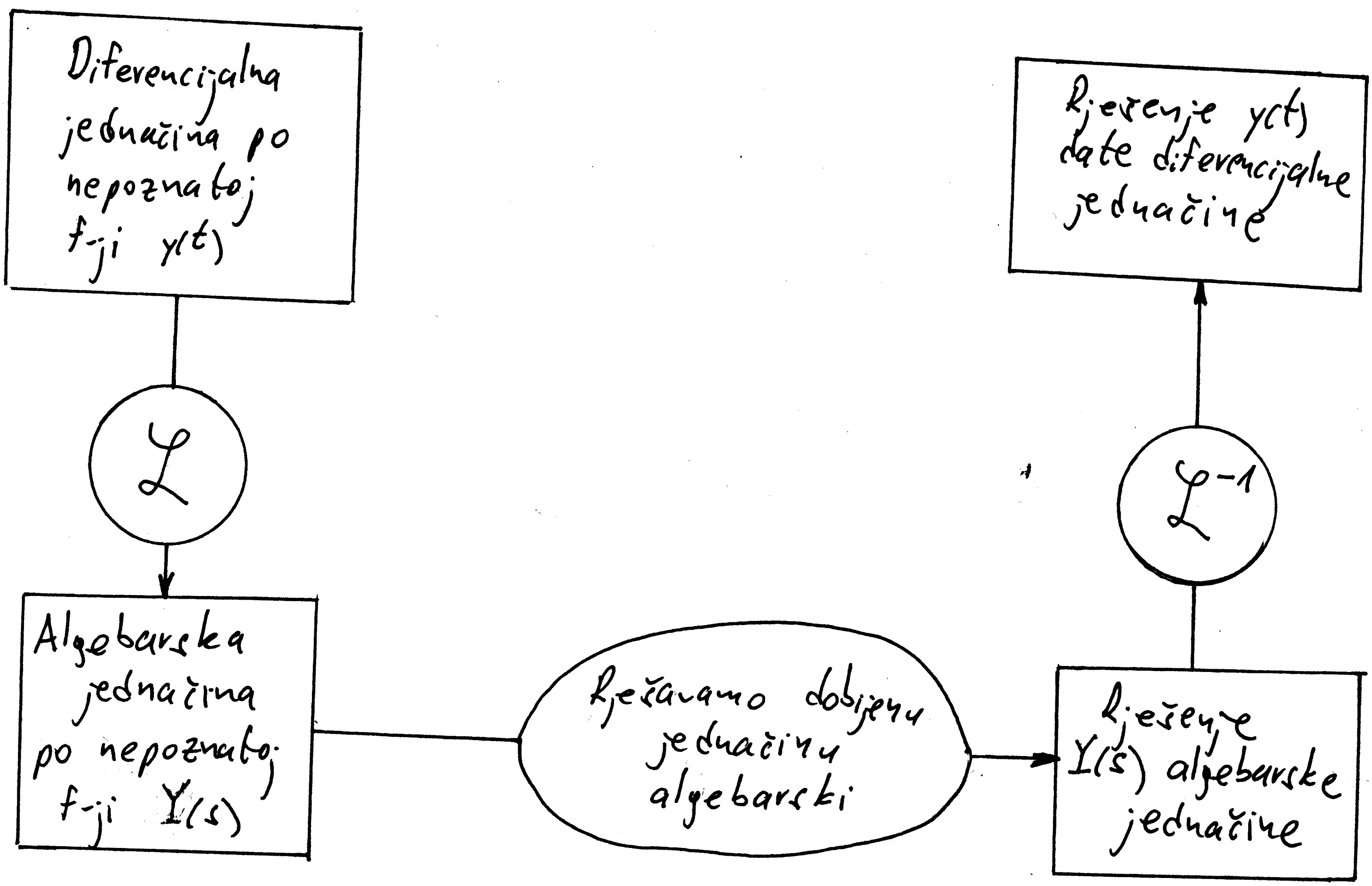
Primjena Laplaceove transformacije pri rješavanju diferencijalnih jednačina

Proceduru za rješavanje diferencijalne jednačine možemo ilustrirati na sljedeći način



Riješiti diferencijalnu jednačinu

$$y'' - y = -t; \quad y(0)=0, \quad y'(0)=1.$$

Rj.

$$y'' - y = -t$$

$$\mathcal{L}\{y''-y\}(s) = \mathcal{L}(-t)$$

$$\mathcal{L}\{y''\}(s) - \mathcal{L}\{y\}(s) = -\mathcal{L}\{t\}(s) \quad \ldots (*)$$

Neka je $Y(s) = \mathcal{L}\{y\}(s)$. Kako je

$$\mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

$$\mathcal{L}\{y''\}(s) = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 1$$

to, prema $(*)$, imamo

$$s^2 Y(s) - 1 - Y(s) = -\frac{1}{s^2} \qquad 1 - \frac{1}{s^2} = \frac{s^2-1}{s^2}$$

$$(s^2-1)\,Y(s) = 1 - \frac{1}{s^2}$$

$$Y(s) = \frac{s^2-1}{s^2(s^2-1)} = \frac{1}{s^2}$$

Kako je $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$ i imamo $Y(s) = \mathcal{L}\{y\}(s)$

to je

$$y(t) = t$$

traženo rješenje.

(#) Riješiti diferencijalnu jednačinu

$$y'' - 2y' + 5y = -8e^{-t}; \quad y(0) = 2, \quad y'(0) = 12.$$

Rj.

$$y'' - 2y' + 5y = -8e^{-t}$$

$$\mathcal{L}\{y'' - 2y' + 5y\}(s) = \mathcal{L}\{-8e^{-t}\}(s)$$

$$\mathcal{L}\{y''\}(s) - 2\mathcal{L}\{y'\}(s) + 5\mathcal{L}\{y\}(s) = -8\,\mathcal{L}\{e^{-t}\}(s) \quad \ldots (1)$$

Neka je $Y(s) = \mathcal{L}\{y\}(s)$. Znamo da

$$\mathcal{L}\{e^{-t}\}(s) = \frac{1}{s+1}$$

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 2,$$

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12$$

Ako ove izraze zamjenimo u (1) imamo

$$(\underline{\underline{s^2Y(s)}} - 2s - 12) - \underline{\underline{2(sY(s)}} - 2) + 5Y(s) = \frac{-8}{s+1}$$

$$(s^2 - 2s + 5)Y(s) = 2s + 8 - \frac{8}{s+1}$$

$$(s^2 - 2s + 5)Y(s) = \frac{2s^2 + 10s}{s+1}$$

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} \quad \boxed{= \frac{3(s-1) + 2\cdot 4}{(s-1)^2 + 2^2} - \frac{1}{s+1}}$$

Ostalo je još da izračunamo inverznu transformaciju racionalne f-je $Y(s)$. Ovo smo već jednom uradili u jednom primjeru iz prethodne lekcije.

$$y(t) = 3e^t\cos 2t + 4e^t\sin 2t - e^{-t}$$

traženo rješenje

(#) Riješiti diferencijalnu jednačinu

$$y''+4y'-5y=te^t; \quad y(0)=1, \quad y'(0)=0.$$

Rj.

$$y''+4y'-5y=te^t$$

$$\mathcal{L}\{y''+4y'-5y\}(s)=\mathcal{L}\{te^t\}(s)$$

$$\mathcal{L}\{y''\}(s)+4\mathcal{L}\{y'\}(s)-5\mathcal{L}\{y\}(s)=\mathcal{L}\{te^t\}(s) \qquad \dots(1)$$

Neka je $Y(s)=\mathcal{L}\{y\}(s)$. Kako je

$$\mathcal{L}\{t\,e^t\}(s)=\frac{1}{(s-1)^2}$$

$$\mathcal{L}\{y'\}(s)=sY(s)-y(0)=sY(s)-1$$

$$\mathcal{L}\{y''\}(s)=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-s$$

to (iz izraza (1)) imamo

$$\left(s^2Y(s)-s\right)+4\left(sY(s)-1\right)-5Y(s)=\frac{1}{(s-1)^2}$$

$$(s^2+4s-5)Y(s)=s+4+\frac{1}{(s-1)^2}$$

$$(s+5)(s-1)Y(s)=\frac{s^3+2s^2-7s+5}{(s-1)^2}$$

$$Y(s)=\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}$$

$$\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}=\frac{A}{s+5}+\frac{B}{s-1}+\frac{C}{(s-1)^2}+\frac{D}{(s-1)^3} \quad \underset{\Longrightarrow}{\text{ZA VJEŽBU}} \quad A=\frac{35}{216},\ B=\frac{181}{216},\ C=-\frac{1}{36},\ D=\frac{1}{6}$$

$$\Rightarrow Y(s)=\frac{35}{216}\left(\frac{1}{s+5}\right)+\frac{181}{216}\left(\frac{1}{s-1}\right)-\frac{1}{36}\left(\frac{1}{(s-1)^2}\right)+\frac{1}{12}\left(\frac{2}{(s-1)^3}\right)$$

$$\Rightarrow y(t)=\frac{35}{216}e^{-5t}+\frac{181}{216}e^t-\frac{1}{36}te^t+\frac{1}{12}t^2e^t$$

traženo rješenje

Rješiti diferencijalnu jednačinu

$$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}; \quad w(\pi) = 2, \quad w'(\pi) = 12.$$

Rj: Da bi koristili metodu Laplasove transformacije, prvo moramo pomjeriti inicijalni uslov na vrijednost $t=0$. Ovo možemo uraditi tako što ćemo uvesti smjenu

$$y(t) := w(t+\pi) \qquad (\, y(0) = w(\pi) \,)$$

Tada

$$y'(t) = w'(t+\pi) \quad ; \quad y''(t) = w''(t+\pi)$$

Zamjenjujući t sa $t+\pi$ u datoj diferencijalnoj jednačini ta jednačina postaje

$$w''(t+\pi) - 2w'(t+\pi) + 5w(t+\pi) = -8e^{\pi-(t+\pi)} = -8e^{-t} \quad \dots(1)$$

Zamjenjujući $y(t) = w(t+\pi)$ u (1), data diferencijalna jednačina postaje

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t}; \quad y(0) = 2, \quad y'(0) = 12.$$

Sad kako je inicijalni uslov dat za $t=0$, možemo iskoristiti metod Laplasove transformacije.

Primjetimo da smo ovu diferencijalnu jednačinu sa istim inicijalnim uslovima već imali, gdje smo pronašli da je njezino rješenje

$$y(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t} \quad \dots(2)$$

Kako je $w(t+\pi) = y(t)$ tada $w(t) = y(t-\pi)$ pa zamjenjujući t sa

$t-\pi$ u (2) dobijemo

$$w(t) = y(t-\pi) = 3e^{t-\pi}\cos(2(t-\pi)) + 4e^{t-\pi}\sin[2(t-\pi)] - e^{-(t-\pi)}$$

$$= 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t - e^{\pi-t}$$

$$\cos(2t-2\pi) = \cos 2t \underbrace{\cos 2\pi}_{=1} + \sin 2t \underbrace{\sin 2\pi}_{=0} = \cos 2t$$

$$\sin(2t-2\pi) = \sin 2t \underbrace{\cos 2\pi}_{=1} - \underbrace{\sin 2\pi}_{=0} \cos 2t = \sin 2t$$

$$w(t) = 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t - e^{\pi-t}$$

traženo rješenje

(#) Riješiti diferencijalnu jednačinu

$$y'' + 2ty' - 4y = 1, \quad y(0) = y'(0) = 0.$$

Rj.

$$y'' + 2ty' - 4y = 1$$

$$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{ty'\}(s) - 4\mathcal{L}\{y\}(s) = \mathcal{L}\{1\}$$

Neka je $Y(s) = \mathcal{L}\{y\}(s)$. Tada

$$\mathcal{L}\{y''\}(s) - 2\mathcal{L}\{ty'\}(s) - 4Y(s) = \frac{1}{s} \qquad \ldots (*)$$

Dalje, kako je

$$\mathcal{L}\{y''\}(s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s)$$

$$\mathcal{L}\{ty'(t)\}(s) = -\frac{d}{ds}\mathcal{L}\{y'\}(s)$$

$$= -\frac{d}{ds}\left(sY(s) - y(0)\right) = -sY'(s) - Y(s)$$

Ako zadnja dva izraza uvrstimo u $(*)$ imamo

$$s^2 Y(s) + 2\left(-sY'(s) - Y(s)\right) - 4Y(s) = \frac{1}{s}$$

$$-2sY'(s) + (s^2 - 6)Y(s) = \frac{1}{s} \qquad /:(-2s)$$

$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = -\frac{1}{2s^2}$$

ovo je linearna diferencijalna jednačina prvog reda

$$Y'(s)+\left(\frac{3}{s}-\frac{s}{2}\right)Y(s)=\frac{-1}{2s^2} \qquad \ldots(1)$$

$$\mu(s)=e^{\int\left(\frac{3}{s}-\frac{s}{2}\right)dx}=e^{\ln s^3-\frac{1}{4}s^2}=s^3e^{-\frac{s^2}{4}} \qquad \ldots(2)$$

Sad ako (1) pomnožimo sa (2)

$$\underbrace{Y'(s)\cdot\mu(s)+\left(\frac{3}{s}-\frac{s}{2}\right)s^3e^{-\frac{s^2}{4}}Y(s)}_{=\frac{d}{ds}(\mu(s)Y(s))}=-\frac{1}{2}\cdot\frac{1}{s^2}\cdot s^3e^{-\frac{s^2}{4}}$$

$$\left(s^3e^{-\frac{s^2}{4}}\right)'_s=3s^2e^{-\frac{s^2}{4}}+s^3\cdot e^{-\frac{s^2}{4}}\cdot\left(-\frac{1}{4}\right)2s=e^{-\frac{s^2}{4}}\left(3s^2-\frac{1}{2}s^4\right)$$

$$=\left(\frac{3}{s}-\frac{s}{2}\right)s^3e^{-\frac{s^2}{4}}.$$

$$\frac{d}{ds}\left(\mu(s)Y(s)\right)=\frac{d}{ds}\left(s^3e^{-\frac{s^2}{4}}Y(s)\right)=-\frac{1}{2}se^{-\frac{s^2}{4}}$$

Sad ako integriramo $\quad s^3e^{-\frac{s^2}{4}}Y(s)=-\frac{1}{2}\underbrace{\int se^{-\frac{s^2}{4}}ds}_{=\int e^{-\frac{s^2}{4}}d\left(-\frac{s^2}{4}\right)}$

$$d\left(-\frac{s^2}{4}\right)=-\frac{1}{4}\cdot 2s\,ds=-\frac{s}{2}ds$$

$$s^3e^{-\frac{s^2}{4}}Y(s)=e^{-\frac{s^2}{4}}+C \Rightarrow Y(s)=\frac{1}{s^3}+C\frac{e^{s^2/4}}{s^3}$$

Sada ćemo iskoristiti sljedeću tvrdnju:

Ako je $f(t)$ po djelovima neprekidna na $[0,\infty)$ i eksponencijalnog reda tada

$$\lim_{s\to\infty}\mathcal{L}\{f\}(s)=0$$

Sad, prema ovoj tvrdnji, ako je $Y(s)$ Laplasova transformacija po djelovima neprekidne f-je eksponencijalnog reda, tada slijedi:

$$\lim_{s\to\infty} Y(s)=0$$

Za $Y(s)=\frac{1}{s^3}+C\frac{e^{\frac{s^2}{4}}}{s^3}$, za kakvo C će vrijediti da $\lim_{s\to\infty} Y(s)=0$?

S obzirom da je $\lim_{s\to\infty}\frac{e^{s^2/4}}{s^3}\left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim_{s\to\infty}\frac{\frac{1}{4}2s\,e^{\frac{s^2}{4}}}{3s^2}=\frac{1}{2}\cdot\frac{1}{3}\lim_{s\to\infty}\frac{e^{\frac{s^2}{4}}}{s}$

$=\ldots=\infty$, imamo, da mora biti $C=0$ (u suprotnom data tvrdnja nije ispunjena.

Prema tome $Y(s)=\frac{1}{s^3}$ i primjenjujući inverznu transformaciju dobijamo

$$y(t)=\frac{1}{2}t^2$$

rješenje date diferencijalne jednačine.

(#) Riješiti diferencijalnu jednačinu

$$x'' - x' - 6x = 0; \quad x(0)=2, \quad x'(0) = -1.$$

R:

$$x'' - x' - 6x = 0$$

$$\mathcal{L}\{x''-x'-6x\}(s) = \mathcal{L}\{0\}(s) \quad \Rightarrow \quad \mathcal{L}\{x''\}(s) - \mathcal{L}\{x'\}(s) - 6\mathcal{L}\{x\}(s) = \mathcal{L}\{0\}(s)$$

Označimo sa $X(s) = \mathcal{L}\{x\}(s)$. S obzirom da je

$$\mathcal{L}\{0\}(s) = 0$$

$$\mathcal{L}\{x'\}(s) = s\mathcal{L}\{x(t)\}(s) - x(0) = sX(s) - 2$$

$$\mathcal{L}\{x''\}(s) = s^2\mathcal{L}\{x(t)\}(s) - sx(0) - x'(0) = s^2X(s) - 2s + 1$$

imamo

$$\left(s^2X(s) - 2s + 1\right) - \left(sX(s) - 2\right) - 6X(s) = 0$$

$$(s^2 - s - 6)X(s) - 2s + 3 = 0$$

$$X(s) = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)}$$

Odredimo koeficijente A i B t.d.

$$\frac{2s-3}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2} \quad /\cdot(s-3)(s+2)$$

$$2s - 3 = A(s+2) + B(s-3)$$

Stavljajući $s=3 \Rightarrow A = \frac{3}{5}$, a za $s=-2 \Rightarrow B = \frac{7}{5}$. Prema tome

$$X(s) = \mathcal{L}\{x(t)\}(s) = \frac{\frac{3}{5}}{s-3} + \frac{\frac{7}{5}}{s+2}.$$ Kako je $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

slijedi:

$$x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$$

(#) Rješiti diferencijalnu jednačinu

$$x'' + 4x = \sin 3t, \qquad x(0) = x'(0) = 0.$$

Rj:

$$x'' + 4x = \sin 3t$$

$$\mathcal{L}\{x'' + 4x\}(s) = \mathcal{L}\{\sin 3t\}(s) \Rightarrow \mathcal{L}\{x''\}(s) + 4\mathcal{L}\{x\}(s) = \frac{3}{s^2+9}$$

Neka je $\mathcal{L}\{x\}(s) = X(s)$. S obzirom da je

$$\mathcal{L}\{x''\}(s) = s^2 X(s) - s x(0) - x'(0) = s^2 X(s)$$

imamo

$$s^2 X(s) + 4X(s) = \frac{3}{s^2+9}$$

$$X(s) = \frac{3}{(s^2+4)(s^2+9)}$$

Odredimo koeficijente A, B, C, D t.d. $\frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$

$$3 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$3 = A(s^3+9s) + B(s^2+9) + C(s^3+4s) + D(s^2+4)$$

$$\begin{aligned} A + C &= 0 \Rightarrow A = -C \\ B + D &= 0 \Rightarrow B = -D \\ 9A + 4C &= 0 \\ 9B + 4D &= 3 \end{aligned}$$

$$\Rightarrow A = 0,\ C = 0,\ B = \frac{3}{5},\ D = -\frac{3}{5}$$

Time smo dobili

$$X(s) = \mathcal{L}\{x(t)\}(s) = \frac{3}{10} \cdot \frac{2}{s^2+2^2} - \frac{1}{5} \cdot \frac{3}{s^2+3^2}$$

Kako je $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$ i $\mathcal{L}\{\sin 3t\}(s) = \frac{3}{s^2+9}$ slijedi da

$$x(t) = \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t.$$

Zadaci za vježbu

1. Korištenjem metode Laplasovih transformacija rješiti diferencijalne jednačine

(a) $y'' - 2y' + 5y = 0$; $y(0) = 2$, $y'(0) = 4$

(b) $y'' + 6y' + 9y = 0$; $y(0) = -1$, $y'(0) = 6$

(c) $w'' + w = t^2 + 2$; $w(0) = 1$, $w'(0) = -1$

(d) $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5$, $y'(0) = -4$.

2. Primjenom Laplace-ove transformacije izvesti izraz za nepoznatu f-ju $Y(s)$ za datu diferencijalnu jednačinu sa datim uslovom

(a) $y'' - 3y' + 2y = \cos t$; $y(0) = 0$, $y'(0) = -1$

(b) $y'' + y' - y = t^3$; $y(0) = 1$, $y'(0) = 0$

(c) $y'' + 5y' - y = e^t - 1$; $y(0) = 1$, $y'(0) = 1$

(d) $y'' - 2y' + y = \cos t - \sin t$; $y(0) = 1$, $y'(0) = 3$.

3. Primjenom metode Laplasove transformacije rješiti diferencijalnu jednačinu trećeg reda

(a) $y''' - y'' + y' - y = 0$; $y(0) = 1$, $y'(0) = 1$, $y''(0) = 3$

(b) $y''' + 3y'' + 3y' + y = 0$; $y(0) = -4$, $y'(0) = 4$, $y''(0) = -2$

4. Rješiti diferencijalne jednačine

(a) $y'' + 3ty' - 6y = 1$; $y(0) = 0$, $y'(0) = 0$.

(b) $ty'' - 2y' + ty = 0$; $y(0) = 1$, $y'(0) = 0$.

Odgovori:

1.

(a) $2e^t\cos 2t + e^t\sin 2t$

(b) $-e^{-3t} + 3t\,e^{-3t}$

(c) $t^2 + \cos t - \sin t$

(d) $\cos t - 4e^{5t} + 8e^{2t}$

2.

(a) $Y(s) = \dfrac{-s^2+s-1}{(s^2+1)(s-1)(s-2)}$

(b) $\dfrac{s^5+s-1}{s^4(s^2+s-1)}$

(c) $\dfrac{s^3+5s^2-6s+1}{s(s-1)(s^2+5s-1)}$

(d) $\dfrac{s^3+s^4+6}{(s^2+1)(s-1)^2}$

3.

(a) $2e^t - \cos t - \sin t$

(b) $(t^2-4)e^{-t}$

4.

(a) $\dfrac{t^2}{2}$

(b) uputa: $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}(t) = \dfrac{1}{2}(\sin t - t\cos t)$

$\cos t + t\sin t + c(\sin t - t\cos t)$, c proizvoljno